MATEMÁTICAS I
Diciembre de 2011

CALCULO DE INTEGRALES

1. $\int x^3 \ dx = \frac{1}{4}x^4 + C$
2. $\int \left(x^2 - 5x^3\right)\sqrt{x} \ dx = \frac{2}{7}x^{\frac{7}{2}} - \frac{10}{9}x^{\frac{9}{2}} + C$
3. $\int (\sqrt{x} + x)(\sqrt{x} - x) \ dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 + C$
4. $\int \frac{2}{x^2+1} \ dx = 2\arctan(x) + C$
5. $\int (3x-4)^5 \ dx = \frac{(3x-4)^6}{18} + C$
6. $\int (1+\operatorname{sen}(x))^3 \cos(x) \ dx = \frac{(1+\operatorname{sen}(x))^4}{4} + C$
7. $\int e^{2\cos(x)} \operatorname{sen}(x) \ dx = -\frac{1}{2}e^{2\cos(x)} + C$
8. $\int \frac{\operatorname{sen}(x)}{\sqrt{1+\cos(x)}} \ dx = -2\sqrt{\cos(x)+1} + C$
9. $\int \frac{1}{2x+1} dx = \frac{1}{2}\ln(2x+1) + C$
10. $\int \frac{3x^2+2}{x^3+2x-3} \ dx = \ln\left(2x + x^3 - 3\right) + C$
11. $\int \tan^2 x \ dx = \tan(x) - x + C$
12. $\int \frac{e^x}{\sqrt{1-e^{2x}}} \ dx = \operatorname{arc\,sen}(e^x) + C$
13. $\int \frac{1}{\operatorname{sen}^2 x} \ dx = -\cot(x) + C$
14. $\int \operatorname{sen}^3(3x)\cos(3x) \ dx = \frac{\operatorname{sen}^4(3x)}{12} + C$
15. $\int \frac{e^x}{1+e^x} \ dx = \ln(e^x + 1) + C$
16. $\int \frac{1}{\sqrt{3x+4}} \ dx = \frac{2}{3}\sqrt{3x+4} + C$
17. $\int \frac{2x}{1+x^4} \ dx = \arctan\left(x^2\right) + C$
18. $\int \frac{1}{(x+2)^3} \ dx = \frac{-1}{2(x+2)^2} + C$
19. $\int 2x \operatorname{sen}\left(x^2+1\right) dx = -\cos\left(x^2+1\right) + C$
20. $\int \operatorname{sen}^2\left(x^2+1\right)\cos\left(x^2+1\right) x dx = \frac{\operatorname{sen}^3\left(x^2+1\right)}{6} + C$

Ejercicio 21.

Calcular
$$\int \ln x \, dx$$
Solución:

La resolvemos por integración por partes:
$$\int u \, du = uv - \int v \, du$$

Hacemos
$$u = \ln x, dv = dx \to v = x, du = \frac{dx}{x}$$
quedando:
$$\int \ln x \, dx = x \ln x - \int x \, \frac{dx}{x} = x \ln x - \int dx = x \ln x - x + C = x(\ln x - 1) + C$$

Ejercicio 22.

Calcular
$$\int x \operatorname{sen} x \, dx$$
Solución:

La resolvemos por integración por partes:
$$\int u \, du = uv - \int v \, du$$

Hacemos
$$u = x, dv = \operatorname{sen} x dx \to v = -\cos x, du = dx$$
quedando:
$$\int x \operatorname{sen} x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \operatorname{sen} x + C$$

Ejercicio 23.

Calcular

$$\int x \cos x \, dx$$

Solución:

La resolvemos por integración por partes:

$$\int u \, du = uv - \int v \, du$$

Hacemos

$$u = x, dv = \cos x dx \to v = \operatorname{sen} x, du = dx$$

quedando:

$$\int x \cos x \, dx = x \operatorname{sen} x - \int \operatorname{sen} x \, dx = x \operatorname{sen} x + \cos x + C$$

Ejercicio 24.

Obtener razonadamente:

$$\int \frac{x+2}{\sqrt{x^2-2x+3}}\,dx$$

Solución:

Vamos a intentar transformar la integral $\int \frac{x+2}{\sqrt{x^2-2x+3}}\,dx$, en una inmediata de tipo potencial:

$$\int f^n(x)f'(x)\,dx = \frac{f^{n+1}(x)}{n+1}+C$$

siendo en este caso $f(x)=x^2-2x+3$ y $n=-1/2$. Por tanto, se resolvería de la siguiente forma:

$$\begin{aligned}
\int \frac{x+2}{\sqrt{x^2-2x+3}}\,dx &= \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2-2x+3}}\,dx = \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2-2x+3}}\,dx = \frac{1}{2}\int \frac{2x-2+6}{\sqrt{x^2-2x+3}}\,dx \\
&= \frac{1}{2}\left(\int \frac{2x-2}{\sqrt{x^2-2x+3}}\,dx + \int \frac{6}{\sqrt{x^2-2x+3}}\,dx\right) \\
&= \frac{1}{2}\int \underbrace{(2x-2)}_{f'(x)}\underbrace{\left(x^2-2x+3\right)^{-\frac{1}{2}}}_{f^n(x)} + \;3\int \frac{1}{\sqrt{x^2-2x+3}}\,dx \\
&= \frac{1}{2}\frac{\left(x^2-2x+3\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + 3\int \frac{1}{\sqrt{x^2-2x+3}}\,dx \\
&= \sqrt{x^2-2x+3} \;+\; 3\underbrace{\int \frac{1}{\sqrt{x^2-2x+3}}\,dx}_{\text{(I)}} \\
&= \sqrt{x^2-2x+3} \;+\; 3\ln\left|\frac{x-1}{\sqrt{2}}+\sqrt{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}\right| + C
\end{aligned}$$

Vamos ahora a resolver la integral casi-inmediata dada en (I) teniendo en cuenta que se cumple que $a=1>0$ y $b^2-4ac<0$:

$$\int \frac{1}{\sqrt{x^2-2x+3}}\,dx = \int \frac{1}{\sqrt{(x-1)^2+2}}\,dx = \int \frac{\frac{1}{\sqrt{2}}}{\sqrt{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}}dx = \ln\left|\frac{x-1}{\sqrt{2}}+\sqrt{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}\right|+C$$

O también se puede resolver así:

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-2x+3}}\,dx &= \int \frac{\sqrt{4}}{\sqrt{4}\sqrt{x^2-2x+3}}\,dx = 2\int \frac{1}{\sqrt{4x^2-8x+12}}\,dx = \\
&= 2\int \frac{1}{\sqrt{(2x-2)^2+8}}\,dx = 2\int \frac{\frac{1}{\sqrt{8}}}{\sqrt{\left(\frac{2x-2}{\sqrt{8}}\right)^2+1}}\,dx = \\
&= \int \frac{\frac{1}{\sqrt{2}}}{\sqrt{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}}\,dx = \ln\left|\frac{x-1}{\sqrt{2}}+\sqrt{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}\right|+C
\end{aligned}$$

Por tanto la solución de la integral es:

$$\int \frac{x+2}{\sqrt{x^2-2x+3}}\,dx = \sqrt{x^2-2x+3}+3\ln\left|\frac{x-1}{\sqrt{2}}+\sqrt{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}\right|+C$$

Ejercicio 25.

Resolver la integral indefinida:

$$\int \frac{dx}{x^2+4}$$

Solución:

Resolvemos la integral indefinida

$$\int \frac{dx}{x^2+4} = \frac{1}{4}\int \frac{dx}{1+(\frac{x}{2})^2} = \frac{2}{4}\int \frac{\frac{1}{2}dx}{1+(\frac{x}{2})^2} = \frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$$

Ejercicio 26.

Resolver la integral indefinida:

$$\int \frac{dx}{x^2-6x+25}$$

Solución:

Resolvemos la integral indefinida

$$\int \frac{dx}{x^2-6x+25} = \int \frac{dx}{(x-3)^2+16} = \int \frac{\frac{1}{16}}{\left[\left(\frac{x-3}{4}\right)^2+1\right]}dx = \frac{4}{16}\int \frac{\frac{dx}{4}}{1+(\frac{x-3}{4})^2} = \frac{1}{4}\arctan(\frac{x-3}{4})+C$$

Ejercicio 27.

Resolver la integral indefinida:

$$\int \frac{3+e^{4x}}{e^{3x}}\ dx$$

Solución:

Resolvemos la integral indefinida

$$\int \frac{3+e^{4x}}{e^{3x}}\ dx = \ cambio[e^x=t \Rightarrow e^x dx = dt] = \int (\frac{t^4+3}{t^3})\frac{dt}{t} =$$

$$= \int\ dt + 3\int t^{-4}\ dt = t + 3\frac{t^{-4+1}}{(-4+1)} + C = \ cambio[t=e^x] = e^x - e^{-3x} + C$$

Ejercicio 28.

Resolver la integral indefinida:

$$\int 2sen(x)cos(x)\ dx$$

Solución:

Resolvemos la integral indefinida

$$\int 2sen(x)cos(x)\ dx = sen^2(x) + C$$

Ejercicio 29.

Resolver la integral indefinida:

$$\int \frac{x^2 + 5x^3}{\sqrt{x}}\ dx$$

Solución:

Resolvemos la integral indefinida

$$\int \frac{x^2 + 5x^3}{\sqrt{x}}\ dx = \int (x^{2-\frac{1}{2}} + 5x^{3-\frac{1}{2}})\ dx = \int (x^{\frac{3}{2}} + 5x^{\frac{5}{2}})\ dx = \frac{2}{5}x^{\frac{5}{2}} + \frac{10}{7}x^{\frac{7}{2}} + C$$

Ejercicio 30.

a) Resolver la integral indefinida:

$$\int \frac{dx}{x^2+x+1}$$

b) Resolver la integral definida:

$$\int_0^2 \frac{dx}{x^2+x+1}$$

Solución:

a) Resolvemos la integral indefinida

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \int \frac{\frac{4}{3}}{\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]\frac{4}{3}}dx = \frac{4}{3}\int \frac{dx}{\left(\frac{2x}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)^2+1}$$

$$= \frac{4}{3}\frac{\sqrt{3}}{2}\int \frac{\frac{2}{\sqrt{3}}}{\left(\frac{2x}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)^2+1}dx = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)+C$$

b) La integral definida será:

$$\int_0^2 \frac{dx}{x^2+x+1} = \left[\frac{2}{\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)\right]_0^2 = \left[\frac{2}{\sqrt{3}}\arctan\left(\frac{2\cdot 2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)-\frac{2}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right)\right] = 0{,}824136$$

Tabla de integrales inmediatas:

$$\text{Tipo potencial} \quad \rightarrow \quad \int (f(x))^n \ f'(x) \ dx = \frac{(f(x))^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\text{Tipo exponencial} \quad \rightarrow \quad \int a^{f(x)} \ f'(x) \ dx = \frac{a^{f(x)}}{\ln a} + C \qquad (a > 0, \ a \neq 1)$$

$$\text{Tipo logarítmico} \quad \rightarrow \quad \int \frac{f'(x)}{f(x)} \ dx = \ln |f(x)| + C \qquad (f(x) \neq 0)$$

$$\text{Tipo trigonométricas} \quad \rightarrow \quad \int \cos(f(x)) \ f'(x) \ dx = \operatorname{sen} f(x) + C$$

$$\rightarrow \quad \int \operatorname{sen}(f(x)) \ f'(x) \ dx = -\cos f(x) + C$$

$$\rightarrow \quad \int \frac{f'(x)}{\cos^2(f(x))} \ dx = \tan f(x) + C$$

$$\rightarrow \quad \int \frac{-f'(x)}{\operatorname{sen}^2(f(x))} \ dx = \cot f(x) + C$$

$$\text{Tipo inversas de trigonométricas} \quad \rightarrow \quad \int \frac{f'(x)}{\sqrt{1-(f(x))^2}} \ dx = \arcsin f(x) + C$$

$$\rightarrow \quad \int \frac{-f'(x)}{\sqrt{1-(f(x))^2}} \ dx = \operatorname{arc}\cos f(x) + C$$

$$\rightarrow \quad \int \frac{f'(x)}{1+(f(x))^2} \ dx = \arctan f(x) + C$$

$$\rightarrow \quad \int \frac{-f'(x)}{1+(f(x))^2} \ dx = \operatorname{arccot} f(x) + C$$

$$\text{Tipo hiperbólicas} \quad \rightarrow \quad \int \cosh(f(x)) \ f'(x) \ dx = \operatorname{senh} f(x) + C$$

$$\rightarrow \quad \int \operatorname{senh}(f(x)) \ f'(x) \ dx = \cosh f(x) + C$$

$$\text{Tipo inversas de hiperbólicas} \quad \rightarrow \quad \int \frac{f'(x)}{\sqrt{1+(f(x))^2}} \ dx = \arg\operatorname{senh}(x) + C$$

$$\rightarrow \quad \int \frac{f'(x)}{\sqrt{(f(x))^2-1}} \ dx = \arg\cosh f(x) + C$$

$$\rightarrow \quad \int \frac{f'(x)}{1-(f(x))^2} \ dx = \arg\tanh f(x) + C \quad |f(x)| < 1$$

$$\rightarrow \quad \int \frac{f'(x)}{1-(f(x))^2} \ dx = \arg\coth f(x) + C \quad |f(x)| > 1$$